# A Distributed Convex Relaxation Approach to Solve the Power Flow Problem

Saeed D. Manshadi, *Member IEEE,* Guangyi Liu, *Senior Member IEEE,* Mohammad E. Khodayar, *Senior Member, IEEE,* Jianhui Wang, *Senior Member IEEE,* Renchang Dai, *Senior Member, IEEE*

***Abstract*— Power flow is a fundamental problem for analyzing the power system. It is to solve a set of equations with quadratic terms. Procuring a reliable solution methodology for this problem is challenging as the feasibility region for this problem is non-convex. Iterative approaches were employed to solve the problem which may fail to provide a solution under certain circumstances such as bad initial point. In this paper, a solution methodology that is capable of providing a reliable solution to the power flow problem is presented. First, by exploiting sparsity in the power network, a convex relaxation for the problem is presented using the first order of the Lasserre hierarchy of moment relaxations. Then a distributed approach using Jacobi-Proximal alternating directions method of multipliers is implemented to efficiently solve the power flow problem. To solve the sub-problems within JP-ADMM approach, the second order of the Lasserre hierarchy of moment relaxations is employed. To illustrate the effectiveness of the proposed approach, several case studies are presented.**

***Index Terms*— Convex relaxation, Jacobi-Proximal ADMM, distributed optimization, Power flow.**

## NOMENCLATURE

Variables and Indices:

| | |
|---|---|
| $c$ | Index for each maximal clique within the network |
| $i, j$ | Index of bus |
| $\mathbf{M}_r^c\{\}$ | Moment relaxation operator for clique $c$ with order $r$ |
| $\mathbf{L}_\gamma\{\}$ | Lifting operator for clique $c$ |
| $\mathbf{q}_{(.)}^c$ | Eigenvector of the moment relaxation matrix associated with maximal clique $c$ |
| $V_i^d$ | Real part of the voltage phasor at bus $i$ |
| $V_i^q$ | Imaginary part of the voltage phasor at bus $i$ |
| $\gamma_{(.)}$ | Lifting variable associated with nonlinear terms |
| $\lambda_{(.)}^c$ | Eigenvalue of the moment relaxation matrix associated with maximal clique $c$ |

Constants and Sets:

| | |
|---|---|
| $B_{ij}$ | The elements of the susceptance matrix |
| $C_c$ | Set of buses connected to clique $c$ |
| $\lvert c\rvert$ | Number of buses within clique $c$ |
| $G_{ij}$ | The elements of the conductance matrix |
| $NB$ | Total number of buses |
| $\mathbf{P}_c$ | The proximal matrix associated with clique $c$ |
| $P_i^{inj}$ | Real power injection at bus $i$ |
| $PQ$ | Set of load buses in the network |
| $PV$ | Set of voltage-controlled buses in the network |
| $Q_i^{inj}$ | Reactive power injection at bus $i$ |
| $Q_i^D$ | Reactive power demand at bus $i$ |
| $Q_i^{G,\min}$ | Minimum reactive power generation for the generation unit connected to bus $i$ |
| $Q_i^{G,\max}$ | Maximum reactive power generation for the generation unit connected to bus $i$ |
| $S$ | Set of slack/reference bus within the network |
| $u_{ij}^c$ | A flag indicating if buses $i$ and $j$ belongs to clique $c$ |
| $V_i'^d$ | The last known measurement for the real part of the voltage phasor at bus $i$ |
| $V_i'^q$ | The last known measurement for the Imaginary part of the voltage phasor at bus $i$ |
| $V_{ref}^d$ | Real part of the voltage phasor at reference bus |
| $V_{ref}^q$ | Imaginary part of the voltage phasor at reference bus |
| $\lvert V_i'\rvert$ | Voltage magnitude at voltage-controlled bus $i$ |
| $V_i^{\max}$ | Maximum voltage magnitude at load bus $i$ |
| $V_i^{\min}$ | Minimum voltage magnitude at load bus $i$ |
| $\alpha$ | A positive damping parameter |
| $\rho$ | A coefficient for convergence of JP-ADMM |
| $y^\psi$ | The vector Lagrangian multiplier at iteration $\psi$ |
| $\psi$ | The index for iterations |
| $\varepsilon$ | A small tolerance for error |
| $\tau_c$ | A constant for proximal term associated with clique $c$ |
| $\chi_c$ | The feasibility region of lifting variables in clique $c$ |

Saeed D. Manshadi is Electrical and Computer Engineering department at San Diego State University, San Diego, CA 92182 (e-mail: smanshadi@sdsu.edu). Mohammad E. Khodayar, and Jianhui Wang are with the Department of Electrical and Computer Engineering, Southern Methodist University, Dallas, TX. USA (e-mail: mkhodayar@smu.edu, jianhui@smu.edu). Guangyi Liu and Renchang Dai are with Global Energy Interconnection Research Institute North America (GEIRI North America or GEIRINA, San Jose, CA, USA (email: guangyi.liu@geirina.net; renchang.dai@geirina.net). This work is supported by State Grid Corporation technology project (SGRIJSKJ(2016)800).

# I. Introduction

Solving power flow is essential for many power system applications including contingency analysis and state estimation. The variabilities imposed by the increase in the penetration level of renewable energy resources further underline the necessity of a reliable framework to solve this problem. To solve the power flow problem a solution to a nonlinear system of equations should be procured and the feasibility of power flow problem is the key to finding such a solution [6]-[8]. Solving a system of nonlinear equations is not trivial. The nonlinear equations form a set of quadratic equations which forms a non-convex feasibility region. To solve the full power flow problem, various iterative approaches are employed. Gauss-Seidel and Newton-Raphson (NR) are the most popular approaches which are employed to solve this problem. However, the non-convex nature of the power flow problem made it difficult for these set of approaches to ensure convergence to a solution. Although the convergence of the NR approach is better than similar iterative approaches, it may fail to provide a solution under some circumstances. In general, several scenarios may happen in solving the system of nonlinear power flow equations. The most trivial one is that a unique solution exists which can be easily procured with an arbitrary initial guess. For the majority of power flow problems, the choice of the initial guess plays an important role in procuring one solution among a set of existing solutions to the problem. However, another scenario is when the iterative approaches fail to render a solution because of a bad initial guess, ill-conditioned power flow or the singularity of the Jacobian matrix during the iterative process [5]-[9]. The last scenario is when the system has no solution. In scenarios where the iterative approach fails to procure a feasible solution, it is hard to determine if the system of equations has no solution or the failure of these approaches is because of numerical issues or inappropriate initiation guess.

To improve the robustness of the iterative approaches, a set of methodologies is proposed in the literature. A direct approach to solve NR is to utilize the lower-upper (LU) factorization which has a considerable computation burden that impedes the large-scale implementations. Krylov subspace introduces a family of methods as presented in [1] to solve the power flow with a finite number of iterations which is at most equal to the number of buses in the network. The Newton-generalized minimal residual method is a member of the family of the Krylov methods employed to solve the power flow problem. Several attempts were made to improve the convergence of Krylov methods. To update the preconditioners for the linearized equations of the next iteration, an adaptive preconditioning scheme is proposed in [2]. Additionally, a flexible inner-outer preconditioner for Newton-generalized minimal residual method is discussed in [3]. In [4], another possible preconditioner for the Krylov method is introduced which utilized the incomplete LU factorization. However, finding proper preconditioning for Krylov subspace-based iterative methods is a challenging task. A continuous Newton's method is enumerated in [5], where the power flow equations are presented by a set of ordinary differential equations.

The motivation of this paper is to provide an efficient solution methodology that not only provides a solution to the power flow problem but also provides insights for possible corrective actions to ensure a feasible solution for an infeasible power flow problem.

Presenting a convex relaxation for the power flow problem is an analytical approach to overcome the short-comings of the iterative approaches [10]. Semi-definite programming relaxation is used to solve different problems in power systems [11]-[14]. This relaxation is usually tighter than a second order cone relaxation, while its scalability is questionable [11]. A sparse formulation is presented in [12] to enable the application of semi-definite relaxation for large-scale problems. To ensure the tightness of the relaxation, a hierarchy of moment relaxations could be leveraged [13]. Although the convergence to a solution which is also feasible for the original non-convex problem is guaranteed, the computation burden of this relaxation made it impractical for large-scale applications. The issue related to scalability of employing higher-orders of the moment relaxation for power system applications is discussed in [14].

In order to address the scalability of solving the power flow problem, a distributed power flow approach based on JP-ADMM algorithm that leverages moment relaxation is presented in this paper. In addition to perturbation, employing a higher order of moment relaxation tightens the presented relaxation. Presenting the distributed framework enables scalability of the presented approach. The main contributions of this paper are listed as follows:

- A distributed convex relaxation framework is proposed, where the second order of moment relaxation is utilized to find a feasible solution. Unlike the iterative approaches, utilizing a convex representation of the power flow problem would determine the feasibility of the power flow problem.
- To facilitate the scalability, the presented relaxation is formulated in sparse form. This will improve the scalability due to the small size of each sub-problem given the small size of the sparse representation of the convex relaxation problem.
- A perturbation function is introduced to render a tighter relaxation that yields a solution among many solutions to the power flow problem. The perturbation enables the system operators to examine the state of the system in proximity of a specific operating point e.g. previous operating point.
- The distributed framework provides an ability to examine the infeasibility of power flow problem to determine a set of corrective actions. Once a sub-problem renders an infeasible solution, the bases of infeasibility within the network can be identified.
- A tightness measure is introduced to examine the merit of proposed relaxation to procuring a solution which is feasible for the original non-convex problem.

This paper is organized as follows: Section II presents the classic power flow problem formulation. Section III presents a solution methodology for the convex relaxation of the problem in a distributed framework. Section IV presents the numerical results to show the effectiveness of the proposed framework to solve the power flow problem, and section V presents the conclusions.

## II. Problem Formulation

The power flow problem using the rectangular representation of complex nodal voltage is given in (1a)-(1e). The input parameters for the power problem are 1) real and reactive power injections for load buses, 2) the voltage magnitude and real power injections of voltage-controlled buses, and 3) the voltage phasor of the slack bus as a reference. Here, the load and voltage-controlled buses are also shown as PQ and PV buses, respectively. The voltage on the slack bus is set to a reference value as stated in (1a) and the solution to the power flow equations would also determine the real and reactive power generation on the slack bus. The real power injection for PV and PQ buses is given in (1b), where the real power injections are given, and the voltage phasors are unknown variables. The reactive power injection for PQ buses is given in (1c), where the reactive power injections are provided, and the voltage phasors are unknown variables. The voltage magnitude for PV buses is controlled as given in (1d). The reactive power generation of the generators connected to PV buses is limited as given in (1e). Once the reactive power limits are reached in these buses, the voltage cannot be controlled anymore. Thus, the reactive power generation is fixed to the capacity of the generation unit connected to the bus and the voltage magnitude will be relaxed meaning that the bus is treated as a PQ bus.

$$V_i^d = V_{ref}^d , V_i^q = V_{ref}^q \qquad \forall i \in S \tag{1a}$$

$$P_i^{inj} = \sum_{j=1}^{NB} \begin{pmatrix} G_{ij}\left(V_i^d V_j^d + V_i^q V_j^q\right) \\ -B_{ij}\left(V_i^d V_j^q - V_i^q V_j^d\right) \end{pmatrix} \qquad \forall i \in \{PV, PQ\} \tag{1b}$$

$$Q_i^{inj} = \sum_{j=1}^{NB} \begin{pmatrix} -B_{ij}\left(V_i^d V_j^d + V_i^q V_j^q\right) \\ -G_{ij}\left(V_i^d V_j^q - V_i^q V_j^d\right) \end{pmatrix} \qquad \forall i \in PQ \tag{1c}$$

$$\sqrt{\left(V_i^d\right)^2 + \left(V_i^q\right)^2} = |V_i| \qquad \forall i \in PV \tag{1d}$$

$$Q_i^{G,\min} \le Q_i^D + \sum_{j=1}^{NB} \begin{pmatrix} -B_{ij}\left(V_i^d V_j^d + V_i^q V_j^q\right) \\ -G_{ij}\left(V_i^d V_j^q - V_i^q V_j^d\right) \end{pmatrix} \le Q_i^{G,\max} \quad \forall i \in PV \tag{1e}$$

The equations with quadratic terms in (1b)-(1c) provide a non-convex feasibility region by nature. Thus, it is challenging to establish a reliable solution methodology that guarantees the procurement of a feasible solution. A distributed optimization framework based on a sparse convex relaxation is formulated in the next section to address this challenge.

## III. Solution Methodology

The power flow problem is formulated as an optimization problem for which any feasible solution is a solution to the power flow problem. To present a distributed convex relaxation for the power flow problem, the second order of moment relaxation is employed. First, the convex relaxation of the power flow problem is formulated in sparse form. Second, a perturbation is used to tighten the relaxation. Finally, the JP-ADMM framework is used to solve the perturbed relaxed sub-problems and yield a solution to the power flow problem. Once the problem is reformulated in the convex relaxed form, not all the solutions provided by the relaxed optimization problem are feasible for the original non-convex problem. However, the feasible solution for the original non-convex power flow problem is among many solutions to the relaxed problem. The detailed description of the presented solution methodology is given as followings.

### A. Convex Relaxation

The non-convex feasibility region of the power flow problem is due to the nonlinear and quadratic terms that include the voltage phasors. The vector of elements of voltage phasors for all buses within the network is presented in (2).

$$x = \begin{bmatrix} V_1^d & \cdots & V_{NB}^d & V_1^q & \cdots & V_{NB}^q \end{bmatrix} \tag{2}$$

In the presented relaxation, the quadratic and nonlinear terms appear in the power flow equations are replaced by lifting terms. As presented in (3), the nonlinear relationships between elements of voltage phasor are dropped and the lifting terms are assembled in a semi-definite matrix. For example, $V_1^q V_2^d$, where $V_1^q \in x$ and $V_2^d \in x$, is a nonlinear term which is replaced by a lifting variable $\gamma_{V_1^q V_2^d} \in X$.

$$xx^T = X \xrightarrow{\text{Relaxation}} X \succeq 0 \tag{3}$$

The semidefinite relaxation is the first order of hierarchy in the Lasserre hierarchy of moment relaxation. Increasing the order of moment relaxation to infinity will guarantee the convergence of the presented relaxation to a solution which is feasible for the original non-convex problem [13]. In this paper, once the solution to the relaxed problem is feasible for the original non-convex problem the relaxation is considered as tight. Theoretically, the relaxation will be tight once the rank of presented semi-definite matrix is one. This means that the relaxation matrix has only one non-zero eigenvalue and therefore, a unique set of values for the element of the voltage phasor vector in (2) can be procured.

In general, there is no guarantee to ensure that the desired tight relaxation is procured by a lower order of moment relaxation. In addition, increasing the order of moment relaxation to infinity is computationally expensive and would make it impractical for the real-world applications. To overcome these challenges, the relaxation is reformulated in sparse form. A perturbation scheme is introduced to tighten the lower orders of the moment relaxation.

To exploit the sparsity in the power network, maximal cliques within the network are defined. Maximal clique by definition is the largest subgraph in which all its nodes are adjacent to each other. It is sufficient to consider the lifting terms within each maximal clique for the power flow problem since all other nonlinear terms are not utilized in the power flow equations. Thus, several small moment relaxation matrices corresponding to maximal cliques are formed. Although, semidefinite programming provides a solution in polynomial time, the computation complexity of solving the problem is $O(n^3)$, where $n$ is the size of each row in the semidefinite matrix. Without using sparsity, $n$ is equal to twice of the number of network buses which makes the problem very expensive to solve. However, by forming maximal cliques and using the sparsity of the network, *n* is equal to twice as the number of buses with the clique which is a relatively small. The sparse form is computationally efficient when higher orders of the hierarchy are employed. If all sparse moment relaxation matrices are tight and render a rank-1 solution, the procured relaxation successfully returns a feasible solution to the original non-convex problem.

To enforce the tightness of the moment relaxation, a perturbation scheme is employed [15]. Perturbation scheme introduces an objective for the relaxed problem which is employed to find the feasible solution for the power flow problem. The choice of the perturbation function will depend on the preferred operating point of the system. Power flow problem may have multiple solutions and the choice for the perturbation function may change the procured feasible solution. As the set of voltages in the network is close to those in the previously known state of the network, here, the perturbation function is defined as to find the solution which is the closest to the last known feasible system condition.

Lifting variables are employed to reformulate the power flow problem in the relaxed form as given in (4). The perturbation function is given in (4a) as the objective of the moment relaxation problem. The real and imaginary parts of the voltage at the slack bus are known as given in (4b). It is assumed that the generation unit connected to the slack bus can provide the required real and reactive power in the network. Reformulating (1b) using the lifting variables, the real power injection balance for PV and PQ buses is given in (4c). The reactive power injection balance for PQ buses is given in (4d). The real and imaginary parts of the voltage on PQ buses are determined by solving the presented optimization problem considering the acceptable voltage limits. Thus, the magnitude of the voltage on PQ buses is also enforced to remain within an acceptable range as shown in (4e). The magnitude of the voltage on PV buses is fixed as shown in (4f). The reactive power generation of PV buses is limited by the physical limits of the generation unit connected to PV bus, as shown in (4g). When the procured solution enforces the reactive power generation to be equal to its limits for PV buses, the PV bus is transformed to a PQ bus with fixed reactive generation. The lifting variables used in the presented relaxed formulation (4a)-(4g), formed a linear matrix inequality (LMI) shown in (4h).

$$\min_{X_c} \quad \sum_i \left( \gamma_{\left(V_i^d\right)^2} - 2V_i'^d V_i^d + \gamma_{\left(V_i^q\right)^2} - 2V_i'^q V_i^q \right) \tag{4a}$$

$$\text{S.t.:} \quad \gamma_{\left(V_i^d\right)^2} = \left(V_{ref}^d\right)^2, \gamma_{\left(V_i^q\right)^2} = \left(V_{ref}^q\right)^2 \qquad \forall i \in S \tag{4b}$$

$$P_i^{inj} = \sum_{j=1}^{NB} \begin{pmatrix} G_{ij}\left(\gamma_{V_i^d V_j^d} + \gamma_{V_i^q V_j^q}\right) \\ -B_{ij}\left(\gamma_{V_i^d V_j^q} - \gamma_{V_i^q V_j^d}\right) \end{pmatrix} \qquad \forall i \in \{PV, PQ\} \tag{4c}$$

$$Q_i^{inj} = \sum_{j=1}^{NB} \begin{pmatrix} -B_{ij}\left(\gamma_{V_i^d V_j^d} + \gamma_{V_i^q V_j^q}\right) \\ -G_{ij}\left(\gamma_{V_i^d V_j^q} - \gamma_{V_i^q V_j^d}\right) \end{pmatrix} \qquad \forall i \in PQ \tag{4d}$$

$$\left(V_i^{\min}\right)^2 \le \gamma_{\left(V_i^d\right)^2} + \gamma_{\left(V_i^q\right)^2} \le \left(V_i^{\max}\right)^2 \qquad \forall i \in PQ \tag{4e}$$

$$\left|V_i'\right|^2 \le \gamma_{\left(V_i^d\right)^2} + \gamma_{\left(V_i^q\right)^2} \le \left|V_i'\right|^2 \qquad \forall i \in PV \tag{4f}$$

$$Q_i^{G,\min} \le Q_i^D + \sum_{j=1}^{NB} \begin{pmatrix} -B_{ij}\left(\gamma_{V_i^d V_j^d} + \gamma_{V_i^q V_j^q}\right) \\ -G_{ij}\left(\gamma_{V_i^d V_j^q} - \gamma_{V_i^q V_j^d}\right) \end{pmatrix} \le Q_i^{G,\max} \quad \forall i \in PV \tag{4g}$$

$$X_c = \begin{pmatrix} 1 & V_i^d & \cdots & V_{|c|}^d & V_i^q & \cdots & V_{|c|}^q \\ V_i^d & \gamma_{\left(V_i^d\right)^2} & \cdots & \gamma_{V_i^d V_{|c|}^d} & \gamma_{V_i^d V_i^q} & \cdots & \gamma_{V_i^d V_{|c|}^q} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ V_{|c|}^d & \gamma_{V_{|c|}^d V_i^d} & \cdots & \gamma_{\left(V_{|c|}^d\right)^2} & \gamma_{V_i^d V_i^q} & \cdots & \gamma_{V_{|c|}^d V_{|c|}^q} \\ V_i^q & \gamma_{V_i^q V_i^d} & \cdots & \gamma_{V_i^q V_{|c|}^d} & \gamma_{\left(V_i^q\right)^2} & \cdots & \gamma_{V_i^q V_{|c|}^q} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ V_{|c|}^q & \gamma_{V_{|c|}^q V_i^d} & \cdots & \gamma_{V_{|c|}^q V_{|c|}^d} & \gamma_{V_{|c|}^q V_i^q} & \cdots & \gamma_{\left(V_{|c|}^q\right)^2} \end{pmatrix} \succeq 0 \qquad \forall c \tag{4h}$$

The presented convex relaxation formulation will render a feasible solution once such a solution exists. However, the sparse representation provides a possibility to employ a distributed framework to solve the relaxed problem. JP-ADMM algorithm is employed in the next subsection to solve the distributed form of this problem.

### B. *Distributed power flow framework*

In any distributed optimization framework, there is a trade-off between the computation burden of solving each sub-problem and the process burden associated with the number of equality constraints that represent consensus among the solutions to the sub-problems. Here, each sub-problem is associated with a maximal clique. The computation burden of solving a clique is low as the number of rows of the associated moment relaxation matrices is small. In JP-ADMM algorithm, first, an initial solution (e.g. the previously measured nodal voltages) is assumed. The main advantage of the JP-ADMM approach is the possibility of solving sub-problems in parallel. Thus, the lifting variables within maximal cliques are updated in parallel until a consensus among those lifting variables that are shared among maximal cliques is reached. For the sake of simplicity, lifting variables are represented as $x_c \in vech(X_c)$, where $vech(.)$ is half-vectorization operator vectorizing the lower triangular part of a symmetric matrix. The convergence of the JP-ADMM approach employed in this paper is discussed in [16]. The algorithm is presented as follows:

**Step 0)** Initialize $x_c^0$ which represents the lifting variables in each clique and $y^0$ which represents the vector of Lagrangian multipliers associated with each equality constraint. Set iteration index to $\psi = 0$.

**Step 1)** Update $x_c^\psi$ for each clique simultaneously using (5). The first term in (5) is the perturbation function with the lifting terms associated with maximal clique *c*. The second term is the

Euclidean distance of the mismatch in the equality constraints of the convex relaxation problem. Here, the distance of the decision variables within a clique with those procured in the previous iteration is minimized. The pre-defined constant ρ sets the weight of this term in the objective, and $y^{\psi}$ is the procured Lagrangian multiplier in previous iteration. The third term is the proximal term to facilitate the convergence. Here $P_c \succeq 0$ is a symmetric and positive semidefinite matrix, where $\|x_c\|_{P_c}^2 := x_c^T P_c x_c$. The choice of $P_c$ facilitates the convergence of this algorithm. Here, the standard proximal method is leveraged where $P_c = \tau_c \mathbf{I}$ and $\tau_c$ as a positive parameter. The feasibility region for the lifting variables within a clique is presented by $\chi_c$.

$$x_c^{\psi+1} \in \underset{x_c \in \chi_c}{\arg\min}\, f_c(x_c) + \left( \begin{array}{c} \frac{\rho}{2} \left\| A_c x_c + \sum_{c' \neq c} A_{c'} x_{c'}^{\psi} - b - \frac{y^{\psi}}{\rho} \right\|_2^2 \\ + \frac{1}{2} \left\| x_c - x_c^{\psi} \right\|_{P_c}^2 \end{array} \right) \quad (5)$$

As several buses are shared between the cliques, the average value of the shared variables determined by each sub-problem is considered. The essence of considering the average value of the variables to reach a consensus among the parallel solutions is presented in [17].

**Step 2)** Check if the consensus is reached by considering the mismatch in the proximal term in (6). If the mismatch is within a predefined tolerance $\varepsilon$ for all cliques, the algorithm is converged and terminated here. Otherwise, proceed to Step 3.

$$\left| x_c^{\psi} - x_c^{\psi+1} \right| < \varepsilon \qquad \forall c \quad (6)$$

**Step 3)** Update the Lagrangian multiplier using (7), where $\alpha$ is a positive damping parameter. Then, update $\psi = \psi + 1$ and proceed to step 1.

$$y^{\psi+1} = y^{\psi} - \alpha\rho\Big(\sum_c A_c x_c^{\psi+1} - b\Big) \quad (7)$$

A detailed description of optimization problem presented in (5) is shown here. The perturbation matrix associated with each clique is presented in (8), where the set of buses that belong to clique *c* is presented by $C_c$.

$$f_c(x_c) = \sum_{i \in C_c} \left( \gamma_{(V_i^d)^2} - 2V_i'^d V_i^d + \gamma_{(V_i^q)^2} - 2V_i'^q V_i^q \right) \quad (8)$$

The equality constraints (4c)-(4d) may include lifting variables from various cliques. These equality constraints are relaxed and embedded in the second term in (5) as expanded in (9). Here, $u_{ij}^c$ is a flag set to 1 if buses *i* and *j* belong to clique *c* and zero otherwise. The expanded form of (9) includes quadratic terms of lifting variables which forms a nonlinear and non-convex problem. Thus, the second order of moment relaxation is employed that use a new set of lifting variables associated with the quadratic terms in (9).

$$\begin{aligned} &\left\| A_c x_c + \sum_{c' \neq c} A_{c'} x_{c'}^{\psi} - b - \frac{y^{\psi}}{\rho} \right\|_2^2 = \\ &\sum_{i \in C_c \cap \{PV, PQ\}} \left( \begin{array}{c} \sum_{j=1}^{NB} u_{ij}^c \left( \begin{array}{c} G_{ij}\left(\gamma_{V_i^d V_j^d} + \gamma_{V_i^q V_j^q}\right) \\ -B_{ij}\left(\gamma_{V_i^d V_j^q} - \gamma_{V_i^q V_j^d}\right) \end{array} \right) \\ \sum_{j=1}^{NB} (1-u_{ij}^c) \left( \begin{array}{c} G_{ij}\left(\gamma^{\psi}_{V_i^d V_j^d} + \gamma^{\psi}_{V_i^q V_j^q}\right) \\ -B_{ij}\left(\gamma^{\psi}_{V_i^d V_j^q} - \gamma^{\psi}_{V_i^q V_j^d}\right) \end{array} \right) - P_i^{inj} - \frac{y_{Pi}^{\psi}}{\rho} \end{array} \right)^2 \\ &+ \sum_{i \in C_c \cap PQ} \left( \begin{array}{c} \sum_{j=1}^{NB} u_{ij}^c \left( \begin{array}{c} -B_{ij}\left(\gamma_{V_i^d V_j^d} + \gamma_{V_i^q V_j^q}\right) \\ -G_{ij}\left(\gamma_{V_i^d V_j^q} - \gamma_{V_i^q V_j^d}\right) \end{array} \right) \\ \sum_{j=1}^{NB} (1-u_{ij}^c) \left( \begin{array}{c} -B_{ij}\left(\gamma^{\psi}_{V_i^d V_j^d} + \gamma^{\psi}_{V_i^q V_j^q}\right) \\ -G_{ij}\left(\gamma^{\psi}_{V_i^d V_j^q} - \gamma^{\psi}_{V_i^q V_j^d}\right) \end{array} \right) - Q_i^{inj} - \frac{y_{Qi}^{\psi}}{\rho} \end{array} \right)^2 \end{aligned} \quad (9)$$

The expanded forms of the terms in (9), using the lifting variables are given in (10) and (11). The lifting terms used in (10)-(11) belong to the second order of moment relaxation problem formulated for a clique. The same lifting terms appear in the expanded form of the third term in (5).

$$\begin{aligned} &\left( \begin{array}{c} G_{ij}\left(\gamma_{V_i^d V_j^d} + \gamma_{V_i^q V_j^q}\right) \\ -B_{ij}\left(\gamma_{V_i^d V_j^q} - \gamma_{V_i^q V_j^d}\right) \end{array} \right)^2 = \\ &G_{ij}^2 \big(\gamma_{(V_i^d)^2 (V_j^d)^2} + 2\gamma_{V_i^d V_j^d V_i^q V_j^q} + \gamma_{(V_i^q)^2 (V_j^q)^2}\big) \\ &+ G_{ij} B_{ij} \left( \begin{array}{c} -2\gamma_{(V_i^d)^2 V_j^d V_j^q} + 2\gamma_{V_i^d (V_i^q)^2 V_i^q} \\ +2\gamma_{V_j^d (V_i^q)^2 V_j^q} - 2\gamma_{V_i^d V_i^q (V_j^q)^2} \end{array} \right) \\ &+ B_{ij}^2 \big(\gamma_{(V_i^d)^2 (V_j^q)^2} - 2\gamma_{V_i^d V_j^d V_i^q V_j^q} + \gamma_{(V_j^d)^2 (V_i^q)^2}\big) \end{aligned} \quad (10)$$

$$\begin{aligned} &\left( \begin{array}{c} -B_{ij}\left(\gamma_{V_i^d V_j^d} + \gamma_{V_i^q V_j^q}\right) \\ -G_{ij}\left(\gamma_{V_i^d V_j^q} - \gamma_{V_i^q V_j^d}\right) \end{array} \right)^2 = \\ &B_{ij}^2 \big(\gamma_{(V_i^d)^2 (V_j^d)^2} + 2\gamma_{V_i^d V_j^d V_i^q V_j^q} + \gamma_{(V_i^q)^2 (V_j^q)^2}\big) \\ &+ G_{ij} B_{ij} \left( \begin{array}{c} +2\gamma_{(V_i^d)^2 V_j^d V_j^q} - 2\gamma_{V_i^d (V_i^q)^2 V_i^q} \\ -2\gamma_{V_j^d (V_i^q)^2 V_j^q} + 2\gamma_{V_i^d V_i^q (V_j^q)^2} \end{array} \right) \\ &+ G_{ij}^2 \big(\gamma_{(V_i^d)^2 (V_j^q)^2} - 2\gamma_{V_i^d V_j^d V_i^q V_j^q} + \gamma_{(V_j^d)^2 (V_i^q)^2}\big) \end{aligned} \quad (11)$$

The monomials associated with the second order moment relaxation is defined as $w_2^c \triangleq x_c$.The *r*th-order moment relaxation for a clique using the lifting term operator is shown in (12). The first order moment relaxation is already defined in (4h). Adopting the notation presented in [13], the monomial associated with the first order moment relaxation is defined by $w_1^c$, i.e. $\mathbf{M}_1^c\{\gamma\} = X_c$.

$$\mathbf{M}_r^c\{\gamma\} \triangleq L_\gamma\{w_r^c w_r^{cT}\} \tag{12}$$

The feasible region of the sub-problem associated with a clique that represented by $\chi_c$ in (5) is formed by (13). The second order moment relaxation formulated for each clique is presented as an LMI (13a). All lifting terms used in (10)-(11) are included in (13a). The inequality constraints (4e)-(4f) can be captured for each clique as the lifting variables exist in them are embedded in the associated moment relaxation matrix. As the second order of moment relaxation is employed here, (13b)-(13e) are the localizing constraints of (4e)-(4f) by adopting the notation in [13]. The last inequality constraint that forms the feasibility region of (5) is the inequality constraint representing the reactive generation limits of the generation units connected to PV buses. The localizing representation of (4g) is provided in (13f)-(13g), where utilized the procured values of the neighboring cliques in the previous iteration.

$$\mathbf{M}_2^c\{\gamma\} \succeq 0 \tag{13a}$$

$$\mathbf{M}_1^c\left\{\left(\gamma_{\left(V_i^d\right)^2} + \gamma_{\left(V_i^q\right)^2} - \left(V_i^{\min}\right)^2\right)\gamma\right\} \succeq 0 \; \forall i \in C_c \cap PQ \tag{13b}$$

$$\mathbf{M}_1^c\left\{\left(\left(V_i^{\max}\right)^2 - \left(\gamma_{\left(V_i^d\right)^2} + \gamma_{\left(V_i^q\right)^2}\right)\right)\gamma\right\} \succeq 0 \; \forall i \in C_c \cap PQ \tag{13c}$$

$$\mathbf{M}_1^c\left\{\left(\left|V_i\right|^2 - \left(\gamma_{\left(V_i^d\right)^2} + \gamma_{\left(V_i^q\right)^2}\right)\right)\gamma\right\} \succeq 0 \quad \forall i \in C_c \cap PV \tag{13d}$$

$$\mathbf{M}_1^c\left\{\left(\gamma_{\left(V_i^d\right)^2} + \gamma_{\left(V_i^q\right)^2} - \left|V_i\right|^2\right)\gamma\right\} \succeq 0 \quad \forall i \in C_c \cap PV \tag{13e}$$

$$\mathbf{M}_1^c\left\{\left(\left(\begin{array}{l}\sum\limits_{j=1}^{NB} u_{ij}^c\begin{pmatrix}-B_{ij}\left(\gamma_{V_i^dV_j^d} + \gamma_{V_i^qV_j^q}\right)\\ -G_{ij}\left(\gamma_{V_i^dV_j^q} - \gamma_{V_i^qV_j^d}\right)\end{pmatrix} + \\ \sum\limits_{j=1}^{NB}(1-u_{ij}^c)\begin{pmatrix}G_{ij}\left(\gamma^{\psi}_{V_i^dV_j^d} + \gamma^{\psi}_{V_i^qV_j^q}\right)\\ -B_{ij}\left(\gamma^{\psi}_{V_i^dV_j^q} - \gamma^{\psi}_{V_i^qV_j^d}\right)\end{pmatrix}\end{array}\right) + Q_i^D - Q_i^{G,\min}\right)\gamma\right\} \succeq 0$$

$$\forall i \in C_c \cap PV \tag{13f}$$

$$\mathbf{M}_1^c\left\{\left(Q_i^{G,\max} - Q_i^D - \left(\begin{array}{l}\sum\limits_{j=1}^{NB} u_{ij}^c\begin{pmatrix}-B_{ij}\left(\gamma_{V_i^dV_j^d} + \gamma_{V_i^qV_j^q}\right)\\ -G_{ij}\left(\gamma_{V_i^dV_j^q} - \gamma_{V_i^qV_j^d}\right)\end{pmatrix} + \\ \sum\limits_{j=1}^{NB}(1-u_{ij}^c)\begin{pmatrix}G_{ij}\left(\gamma^{\psi}_{V_i^dV_j^d} + \gamma^{\psi}_{V_i^qV_j^q}\right)\\ -B_{ij}\left(\gamma^{\psi}_{V_i^dV_j^q} - \gamma^{\psi}_{V_i^qV_j^d}\right)\end{pmatrix}\end{array}\right)\right)\gamma\right\} \succeq 0$$

$$\forall i \in C_c \cap PV \tag{13g}$$

Each of the moment relaxation matrices presented in (13b)-(13g) are the localizing matrices associated with the constraints that form the feasible region of (5). For the sake of clarity, the localizing constraint (13b) is expanded in (14). Here, it is assumed that the buses $i$ and $j$ are the only two members of a clique. All lifting variables used in (14) are defined in (13a).

*C. Discussion on the feasibility of the relaxed power problem*

The presented distributed framework using JP-ADMM will converge to a solution for the relaxed problem. The solution procured will be feasible for the original power flow problem when the rank of all first-order moment relaxation matrices associated with the maximal cliques is 1. However, reaching an exact rank 1 is impractical as the LMIs utilized in the presented distributed framework are solved by interior-point algorithms. Thus, a tightness measure is used here to determine the quality of the solution. The tightness of the procured solution for each clique is presented as the ratio of the largest eigenvalue of the associated first-order moment relaxation matrix to the second largest eigenvalue of that matrix as shown in (15). Here, $TR_c$ is the tightness measure for each maximal clique $c$. Larger ratio means the rank of the associated moment relaxation matrix is closer to 1.

$$TR_c = \log\left(\frac{\lambda_{|c|}^c}{\lambda_{|c|-1}^c}\right) \tag{15}$$

Using the Cholesky decomposition, the elements of the voltage phasor for each clique can be procured using (16). If the second largest eigenvalue is much smaller than the largest eigenvalue, a unique set of voltages can be procured. Here, the vector of voltages is a span of multiplying non-dominated eigenvalues to their associated eigenvectors.

$$\begin{pmatrix}
\gamma_{{V_i^d}^4} + \gamma_{{V_i^d}^2{V_i^q}^2} - {V_i^{\min}}^2\gamma_{{V_i^d}^2} & \gamma_{{V_i^d}^3V_j^d} + \gamma_{V_i^dV_j^d{V_i^q}^2} - {V_i^{\min}}^2\gamma_{V_i^dV_j^d} & \gamma_{{V_i^d}^3V_i^q} + \gamma_{V_i^d{V_i^q}^3} - {V_i^{\min}}^2\gamma_{V_i^dV_i^q} & \gamma_{{V_i^d}^3V_j^q} + \gamma_{V_i^d{V_i^q}^2V_j^q} - {V_i^{\min}}^2\gamma_{V_i^dV_j^q} \\
\gamma_{{V_i^d}^3V_j^d} + \gamma_{V_i^dV_j^d{V_i^q}^2} - {V_i^{\min}}^2\gamma_{V_i^dV_j^d} & \gamma_{{V_i^d}^2{V_j^d}^2} + \gamma_{{V_j^d}^2{V_i^q}^2} - {V_i^{\min}}^2\gamma_{{V_i^q}^2} & \gamma_{{V_i^d}^2V_j^dV_i^q} + \gamma_{V_j^d{V_i^q}^3} - {V_i^{\min}}^2\gamma_{V_j^dV_i^q} & \gamma_{{V_i^d}^2V_j^dV_j^q} + \gamma_{V_j^d{V_i^q}^2V_j^q} - {V_i^{\min}}^2\gamma_{V_j^dV_j^q} \\
\gamma_{{V_i^d}^3V_i^q} + \gamma_{V_i^d{V_i^q}^3} - {V_i^{\min}}^2\gamma_{V_i^dV_i^q} & \gamma_{{V_i^d}^2V_j^dV_i^q} + \gamma_{V_j^d{V_i^q}^3} - {V_i^{\min}}^2\gamma_{V_j^dV_i^q} & \gamma_{{V_i^d}^2{V_i^q}^2} + \gamma_{{V_i^q}^4} - {V_i^{\min}}^2\gamma_{{V_i^q}^2} & \gamma_{{V_i^d}^2V_i^qV_j^q} + \gamma_{{V_i^q}^3V_j^q} - {V_i^{\min}}^2\gamma_{V_i^qV_j^q} \\
\gamma_{{V_i^d}^3V_j^q} + \gamma_{V_i^d{V_i^q}^2V_j^q} - {V_i^{\min}}^2\gamma_{V_i^dV_j^q} & \gamma_{{V_i^d}^2V_j^dV_j^q} + \gamma_{V_j^d{V_i^q}^2V_j^q} - {V_i^{\min}}^2\gamma_{V_j^dV_j^q} & \gamma_{{V_i^d}^2V_i^qV_j^q} + \gamma_{{V_i^q}^3V_j^q} - {V_i^{\min}}^2\gamma_{V_i^qV_j^q} & \gamma_{{V_i^d}^2{V_j^q}^2} + \gamma_{{V_i^q}^2{V_j^q}^2} - {V_i^{\min}}^2\gamma_{{V_j^q}^2}
\end{pmatrix} \succeq 0 \tag{14}$$

$$[V_1^d,...,V_{|c|}^d,V_1^q,...,V_{|c|}^q]=\lambda_{|c|}^c \mathbf{q}_{|c|}^{c\ T}+\lambda_{|c|-1}^c \mathbf{q}_{|c|-1}^{c\ T}+....\lambda_1^c \mathbf{q}_1^{c\ T} \quad (16)$$

## IV. Numerical Results

To show the merit of the proposed approach, several test case studies are analyzed. The data for these cases can be found in [9], [18], [19]. To show the effectiveness of the proposed approach in procuring a solution for the ill-conditioned power flow problems, Case43-ill and Cas13-ill presented in [9] are studied. To justify the scalability of the presented algorithm with the increase in the size of the power network, the sparse structures of these cases are analyzed. The total number of cliques, the average size of cliques, the average size of each clique, the maximum size of cliques, and the number of shared variables are presented in Table I.

TABLE I
THE SPARSE STRUCTURE OF VARIOUS TEST CASES

| Test case | Total number of cliques | The average size of a clique | The max. size of a clique | Total number of shared variables | Solution Time [s] |
|---|---|---|---|---|---|
| 4gs | 4 | 2 | 2 | 8 | 0.58 |
| Case9 | 9 | 2 | 2 | 18 | 0.65 |
| Case14 | 12 | 2.41 | 3 | 20 | 0.63 |
| IEEE 30 | 29 | 2.2 | 3 | 44 | 0.82 |
| Case 33 | 32 | 2 | 2 | 56 | 0.88 |
| Case57 | 62 | 2.14 | 3 | 106 | 1.28 |
| IEEE 118 | 141 | 2.14 | 4 | 198 | 2.12 |
| Case 200 | 223 | 2.05 | 3 | 252 | 4.07 |
| Case2383 | 2836 | 2.01 | 3 | 3732 | 35.11 |
| Case13-ill | 6 | 2.33 | 3 | 10 | 0.67 |
| Case43-ill | 42 | 2 | 2 | 44 | 0.77 |

The average size of formed cliques is small for all cases. The number of shared variables has direct correlation with the number of cliques. In addition, with the increase in the number of buses, the solution time is not dramatically increased. The correlation between the sizes of various case studies with the solution time to solve the problem is shown in Table I. As it is shown, with the exponential increase in the size of the network, the solution time does not increase proportionally and therefore, the presented solution methodology could be used for real-world power flow problems.

To show the exactness of the solution procured by the presented algorithm the introduced tightness ratio is quantified. The average, standard deviation, smallest and largest values for the tightness measures for the cliques in various test cases are provided in Table II. The tightness measure in all case studies reached to a level that the procured solution could be considered as feasible for the original non-convex power flow problem. The standard deviation of the tightness measures shed lights on the overall range of tightness ratios for various cliques in the test case studies. For example, a very small standard deviation in Cases4gs, Case9, Case13-ill and Case 14 illustrates that the tightness ratios of all cliques are in a close neighborhood of their average values. The minimum and maximum tightness ratios reveal the best- and worst-quality solutions. A very low tightness ratio indicates that there are some cliques within the network that may not yield a tight enough solution. The closeness of the minimum tightness ratio to the average tightness ratio with a small standard deviation reveals that the tightness measure for all cliques is almost equal to the average tightness measure (e.g. Case 33). If the maximum tightness ratio is significantly larger than the average value with small standard deviation, there are few cliques that are exceptionally tighter than the average tightness ratio. An example of this situation is in IEEE-118 bus system. If the minimum tightness ratio is significantly smaller than the average with small standard deviation, there are few cliques that have exceptionally looser relaxation gap than the average tightness ratio (e.g. Case 200 and Case 2383). The relaxation might be tightened by some perturbation in a clique with exceptionally small tightness ratio.

TABLE II
THE ANALYSIS OF TIGHTNESS FOR VARIOUS TEST CASES

| Test case | Average $TR_c$ | Standard deviation of $TR_c$ | Minimum $TR_c$ | Maximum $TR_c$ |
|---|---|---|---|---|
| Case 4gs | 8.39 | 0.07 | 8.28 | 8.47 |
| Case9 | 8.62 | 0.11 | 8.48 | 8.76 |
| Case14 | 7.92 | 0.40 | 7.26 | 8.53 |
| IEEE 30 | 7.14 | 1.31 | 3.89 | 8.35 |
| Case 33 | 7.46 | 0.05 | 7.35 | 7.59 |
| Case57 | 6.11 | 1.69 | 2.05 | 7.72 |
| IEEE 118 | 7.79 | 1.16 | 2.97 | 8.87 |
| Case 200 | 6.25 | 1.03 | 2.19 | 7.07 |
| Case 2383 | 5.51 | 0.50 | 2.45 | 6.14 |
| Case13-ill | 7.59 | 0.52 | 6.86 | 8.44 |
| Case43-ill | 5.45 | 0.43 | 4.12 | 5.81 |

The ultimate solution to the power flow problem is the voltage profile. The voltage profile of the procured power flow should be within an acceptable range to represent a physically meaningful solution. The average, standard deviation, minimum as well as the maximum values for the voltage magnitudes and angles of the buses for each test case are given in Tables III and IV, respectively. As shown in Table III, all voltages are close to 1 p.u. with a relatively small standard deviation as expected. The minimum and maximum voltage magnitudes are also within the desired limits. The voltage angles are also very small which indicates procuring of a set of valid solutions for various test cases as it is desired to keep the difference in voltage angles a small number for the stability of the system.

TABLE III
THE VOLTAGE MAGNITUDE FOR VARIOUS TEST CASES

| Test case | Average voltage magnitude [p.u] | Standard deviation of voltage magnitude [p.u.] | Minimum voltage magnitude [p.u.] | Maximum voltage magnitude [p.u.] |
|---|---|---|---|---|
| Case4gs | 1.0005 | 0.0205 | 0.9763 | 1.0207 |
| Case9 | 0.9895 | 0.0068 | 0.9810 | 1.0032 |
| Case14 | 1.0415 | 0.0161 | 1.0110 | 1.0600 |
| IEEE 30-bus | 1.0004 | 0.0148 | 0.9780 | 1.0388 |
| Case 33 | 0.9485 | 0.0303 | 0.9131 | 1.0000 |
| Case57 | 1.0251 | 0.0272 | 0.9669 | 1.0600 |
| IEEE 118 | 0.9821 | 0.0205 | 0.9401 | 1.0489 |
| Case 200 | 1.0287 | 0.0249 | 0.98 | 1.100 |
| Case2383 | 1.00093 | 0.0264 | 0.9210 | 1.1100 |
| Case13-ill | 1.0234 | 0.0325 | 0.9628 | 1.0589 |
| Case 43-ill | 1.0153 | 0.0352 | 0.9457 | 1.1000 |

TABLE IV
THE VOLTAGE ANGLE FOR VARIOUS TEST CASES

| Test case | Average voltage magnitude [deg.] | Standard deviation of voltage magnitude [deg.$^2$] | Minimum voltage magnitude [deg.] | Maximum voltage magnitude [deg.] |
|---|---|---|---|---|
| Case 4gs | 1.2406 | 0.7636 | 0.2815 | 1.9443 |
| Case9 | 1.3872 | 1.5629 | 0.0145 | 3.8385 |
| Case14 | 12.4093 | 4.2081 | 0.1512 | 18.8267 |
| IEEE 30-bus | 1.8474 | 1.1174 | 0.0589 | 4.7442 |
| Case 33 | 0.2159 | 0.1555 | 0.0157 | 0.4978 |
| Case57 | 89.1508 | 4.0331 | 76.6198 | 96.1728 |
| IEEE 118 | 22.2289 | 7.7311 | 8.3984 | 40.6531 |
| Case 200 | 18.5639 | 6.2415 | 0.4563 | 31.0847 |
| Case2383 | 10.7944 | 7.5793 | 0.3995 | 27.3941 |
| Case 13-ill | 4.7984 | 2.3888 | 1.7359 | 10.2060 |
| Case 43-ill | 5.8980 | 3.5884 | 0.8998 | 18.6341 |

The procured voltage profiles using the proposed approach and NR method are shown in Fig. 1-7 for several cases. Here, the constraint (4b) is relaxed in Fig. 1-4 to enable distinguishing between the solutions procured using the proposed approach and the NR method. The voltage profile in Fig. 5 demonstrates the same voltages procured using both approaches as (4b) is enforced. It is worth noting that the NR approach is not capable of rendering a solution for the ill-conditioned cases of 13-bus and 43-bus systems. Thus, the voltage profile that is given in Fig. 6 and Fig. 7 only includes the solution rendered by the proposed approach.

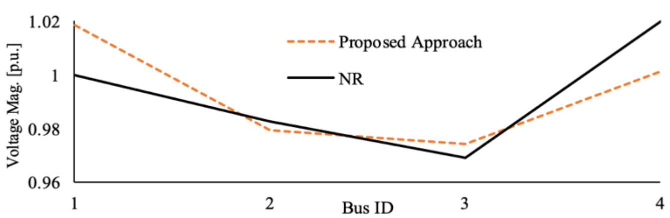

Fig. 1. The procured voltage profiles using the proposed approach and NR method for Case4gs

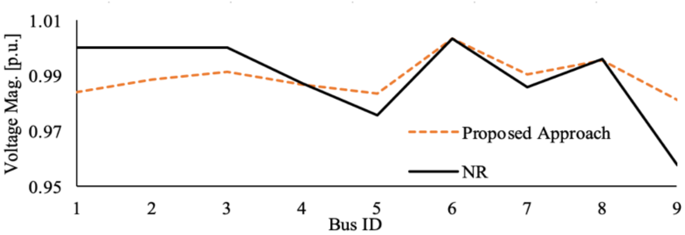

Fig. 2. The procured voltage profiles using the proposed approach and NR method for Case 9

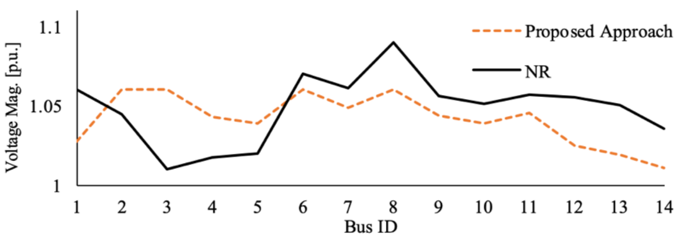

Fig. 3. The procured voltage profiles using the proposed approach and NR method for Case 14

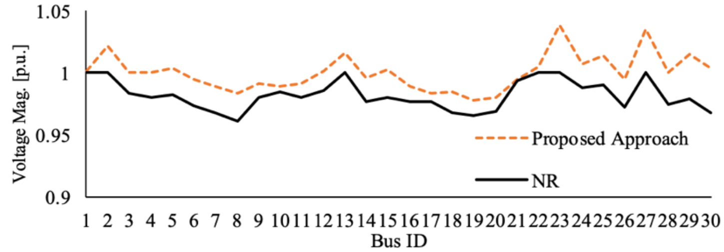

Fig. 4. The procured voltage profiles using the proposed approach and NR method for IEEE 30-bus system

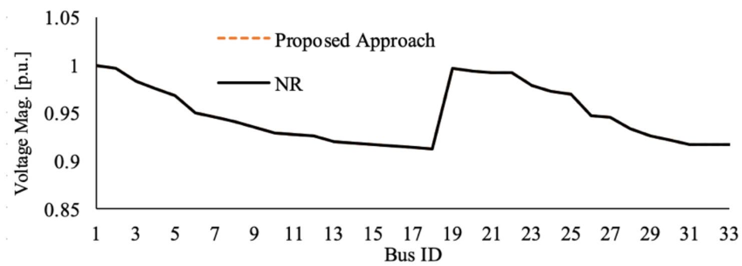

Fig. 5. The procured voltage profiles using the proposed approach and NR method for case33

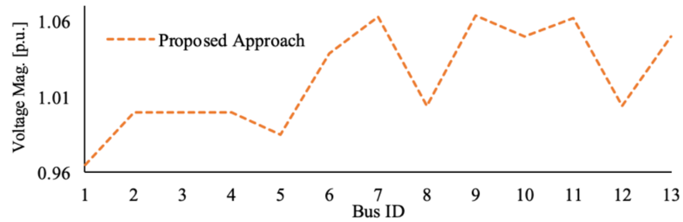

Fig. 6. The procured voltage profiles using the proposed approach and NR method for case13-ill

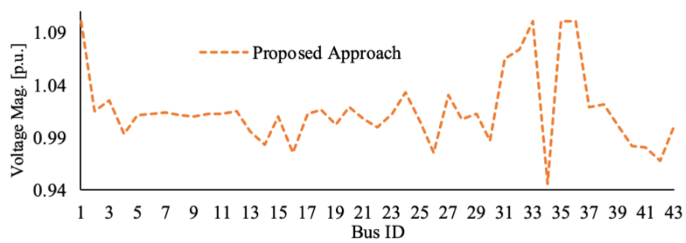

Fig. 7. The procured voltage profiles using the proposed approach and NR method for Case43-ill

Obtaining the loss of the system is another objective for solving the power flow problem. The values of system loss for the presented cases are given in Table V. When constraint (4b) is relaxed, the values of network loss procured using the proposed approach for Case4gs, Case9, and Case14 are smaller than those provided by the NR method . The network loss for Case33 is the same for the proposed method and NR when constraint (4b) is enforced. Thus, the proposed approach is capable of providing a solution with less network loss compared to NR method.

TABLE V
THE COMPARISON OF THE PERCENT OF LOSSES FOR VARIOUS TEST CASES

| Method | Case4gs | Case9 | Case14 | Case33 | Case13-ill | Case43-ill |
|---|---|---|---|---|---|---|
| NR | 0.99 | 1.58 | 5.17 | 5.45 | NA | NA |
| Proposed approach | 0.99 | 0.70 | 0.96 | 5.45 | 0.94 | 1.21 |

## V. Conclusion

This paper presented a convex relaxation framework that provides a feasible solution for the power flow problem in polynomial time. The Jacobi-Proximal alternative direction method of multipliers algorithm is employed to ensure convergence among several sub-problems. In each sub-problem, the second-order moment relaxation is employed to reformulate the power flow problem in relaxed form. This relaxation which provides a tighter relaxation compared to the first-order moment relaxation is further tightened by introducing a perturbation function. This function seeks the closest solution to a specific solution to the power flow problem. An example of such a solution is the last known power flow solution of the network. While employing higher orders of moment relaxation provides a tighter relaxation with huge computation burden, leveraging lower order of moment relaxation will not notably increase the computation burden. Using perturbation provides a solution for the presented relaxed problem which is also feasible for the original non-convex power flow problem. The effectiveness of the presented approach for several test cases is presented. The scalability of the presented algorithm is illustrated, where for a network with thousands of buses, the presented relaxation is tight and render the solution to the original non-convex power flow problem. The potential future work is to utilize the proposed solution method as a fundamental analysis tool for various power system problems. For example, the presented framework can be utilized in state estimation and contingency analysis problems. Another potential future direction for the presented study is to focus on accelerating the algorithm to decreasing the solution time.

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