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# A Tight Convex Relaxation for the Natural Gas Operation Problem

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***Abstract*— With the increase in the investment in the gas-fired electricity generation technology, capturing the operation constraints of the natural gas network in the electricity and natural gas operation problems becomes more crucial. The non-convexity in the feasibility region formed by natural gas network constraints will impede achieving the global solution for these problems. This letter proposed an algorithm to procure a tight and tractable convex relaxation for the natural gas network constraints that can be leveraged in the electricity and/or natural gas network operation problems. The merit of the proposed algorithm is illustrated using a case study and the efficiency of the proposed formulation is compared with mixed integer linear and semidefinite programming formulations.**

***Index Terms*— Convex relaxation, natural gas network, operation economics**

## I. Introduction

THE increase in the installed capacity of the renewable resources and the reduction in the price of the natural gas increase the installed capacity of the gas-fired generation units. This transformation in the generation portfolio will affect the nodal pressure in the natural gas network. The nonlinear relationship between the nodal pressure and the pipeline flow is enforced by Weymouth constraint. This constraint forms a nonconvex feasible region for the natural gas operation problem and mixed-integer linear programming approximation [1] and piecewise linear approximations [2] are employed to convexify the feasible region. Increasing the number of piecewise linear approximations improves the accuracy of the convex representation of the feasible region. Iterative approaches such as Newton-based and interior point methods were applied to find a solution for the electricity operation problem with natural gas network constraints [3],[4]; however, these approaches often suffer from dependence to the initial point and convergence to local solutions. Furthermore, capturing the bi-directional flow of the natural gas in the operation problem is challenging using these approaches and the direction of the natural gas flow is considered as a priori. This letter proposed a tight convex relation of the natural gas operation problem to achieve a global solution with enhanced accuracy. A tractable algorithm is presented to ensure the tightness as well as the validity of the convex representation for the natural gas operation planning problem. The effectiveness of the proposed approach is evaluated for a sample natural gas network. To illustrate the merit of the proposed approach, the proposed tight convex relaxation is compared with the case in which the problem is formulated as 1) a MILP problem and solved by branch and bound approach and 2) SDP relaxed problem solved by primal-dual interior point method.

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## II. Problem Formulation

The problem formulation for the natural gas operation planning problem is given in (1).

$$\min \textstyle\sum_s \xi_s \cdot v_s \tag{1a}$$
$$\underline{\pi} \le \pi_j \le \overline{\pi} \qquad \forall j \tag{1b}$$
$$-f_{j,o}^{max} \le f_{j,o} \le f_{j,o}^{max} \qquad \forall j, \forall o \tag{1c}$$
$$\underline{v}_s \le v_s \le \overline{v}_s \qquad \forall s \tag{1d}$$
$$\textstyle\sum_s A_j^s \cdot v_s = d_j + \sum_{o \in J_j} f_{j,o} \qquad \forall j \tag{1e}$$
$$\begin{cases} f_{j,o} = sgn(\pi_j, \pi_o) \cdot C_{j,o} \sqrt{|\pi_j - \pi_o|} \\ sgn(\pi_j, \pi_o) = \begin{cases} 1 & \pi_j \ge \pi_o \\ -1 & \pi_j < \pi_o \end{cases} \end{cases} \qquad \forall j, \forall o \tag{1f}$$
$$\pi_j \le \Gamma_{j,o} \cdot \pi_o \qquad \Gamma_{j,o} \ge 1 \tag{1g}$$

The total cost of withdrawn natural gas is minimized as shown in (1a), where $\xi_s$ is the marginal cost of supply at the resource $s$ and $v_s$ is a decision variable for the volume of natural gas withdrawn from resource $s$ [5]. Other objectives including minimizing the power consumption of the compressors could be considered for the natural gas operation problem [6]. The pressure limits at junctions are illustrated in (1b). Here, $\pi_j$ is the decision variable for the squared pressure at junction $j$, while $\underline{\pi}$ and $\overline{\pi}$ are the minimum and maximum squared nodal pressure. Additionally, the capacity limits of natural gas flow in pipelines are given in (1c), where $f_{j,o}$ is the natural gas flow in the pipeline, and $f_{j,o}^{max}$ is the capacity of the pipeline connecting junctions $j$ and $o$. The capacity of natural gas supply at source junctions is given in (1d), where the maximum and minimum limits for the withdrawn natural gas at the source $s$ is shown by $\overline{v}_s$ and $\underline{v}_s$, respectively. The supply/demand balance at junction $j$ is presented in (1e), where $A_j^s$ is an incidence matrix for the junction-resource, $d_j$ is the demand at junction $j$, and $J_j$ is the set of junctions connected to junction $j$. Despite the assumptions made in [5], the direction of flow in the pipeline is not a priori. The non-convex Weymouth constraint that represents the dependence of the natural gas flow to the pressures at the connected junctions is given in (1f). Here $C_{j,o}$ is the constant for the pipeline connecting junctions $j$ and $o$, and $sgn(\pi_j, \pi_o)$ is determined by the difference among the pressure

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at junctions $j$ and $o$. The compression in the natural gas network is represented by (1g), where $\Gamma_{j,o}$ is the maximum compression ratio of the compressor [5].

The formulation (1) is a nonlinear and nonconvex problem because of the set of nonconvex constraints (1f). The set of constraints (1f) are reformulated to (2) in which $\overline{u}_{j,o}$is a binary variable. If $\pi_j \geq \pi_o$ then $sgn(\pi_j, \pi_o) = 1$ and $\overline{u}_{j,o} = 0$. Otherwise, $\overline{u}_{j,o} = 1$ and $sgn(\pi_j, \pi_o) = -1$. Constraint (2) is further relaxed by introducing the lifting variables.

$$f_{j,o} = C_{j,o}\left(\sqrt{|\pi_j - \pi_o|} - 2\overline{u}_{j,o}\sqrt{|\pi_j - \pi_o|}\right) \quad (2)$$

## III. Solution Methodology

This section presents a novel algorithm to procure a tight convex relaxation for the problem (1) utilizing the lower orders of the moment relaxation compared to the current approaches [7]. The non-convex optimization problem (1) can be reformulated into a convex relaxed form (3). Here, a sparse first order moment matrix for the nonlinear variables associated with pipelines connected to junction $j$ is presented in (3f), where $\gamma_{(.)}$ represents the lifting variables that replace the nonlinear terms in the natural gas operation problem. The subscript indicates the nonlinear terms that is relaxed using the lifting variables. For example, lifting variable $\gamma_{\sqrt{|\pi_j-\pi_o|}}$ replaces the nonlinear term $\sqrt{|\pi_j - \pi_o|}$. All lifting variables are presented in the moment relaxation matrix given in (3f). Here, (1c) is rewritten as (3c) as $f_{j,o}$ is replaced by the function of junction pressures given in (2) and the introduced lifting variables. The inequality constraint (3d) is written using (1d) in which $v_s$ is represented as a function of demand $d_j$ and $f_{j,o}$ using (1e) and $f_{j,o}$ is formulated using lifting variables in (2).

$$\min \textstyle\sum_s \xi_s \cdot \left[\sum_j A_j^s \cdot \left(d_j + \sum_{o\in J_j} C_{j,o}\left(\gamma_{\sqrt{|\pi_j-\pi_o|}} - 2\gamma_{\overline{u}_{j,o}\sqrt{|\pi_j-\pi_o|}}\right)\right)\right] \quad (3a)$$

$$\underline{\pi} \leq \pi_j \leq \overline{\pi} \qquad \forall j \quad (3b)$$

$$-f_{j,o}^{max} \leq C_{j,o}\left(\gamma_{\sqrt{|\pi_j-\pi_o|}} - 2\gamma_{\overline{u}_{j,o}\sqrt{|\pi_j-\pi_o|}}\right) \leq f_{j,o}^{max} \quad \forall j,o \quad (3c)$$

$$\textstyle\sum_s A_j^s \cdot \underline{v}_s \leq d_j + \sum_{o\in J_j} C_{j,o}\left(\gamma_{\sqrt{|\pi_j-\pi_o|}} - 2\gamma_{\overline{u}_{j,o}\sqrt{|\pi_j-\pi_o|}}\right) \leq \sum_s A_j^s \cdot \overline{v}_s \qquad \forall j \quad (3d)$$

$$\pi_j \leq \Gamma_{j,o} \cdot \pi_o \qquad \Gamma_{j,o} \geq 1 \quad (3e)$$

$$\begin{bmatrix} 1 & \gamma_{\overline{u}_{j,o}} & \gamma_{\sqrt{|\pi_j-\pi_o|}} \\ \gamma_{\overline{u}_{j,o}} & \gamma_{\overline{u}_{j,o}} & \gamma_{\overline{u}_{j,o}\sqrt{|\pi_j-\pi_o|}} \\ \gamma_{\sqrt{|\pi_j-\pi_o|}} & \gamma_{\overline{u}_{j,o}\sqrt{|\pi_j-\pi_o|}} & \pi_j - \pi_o + 2\gamma_{-\overline{u}_{j,o}(\pi_j-\pi_o)} \end{bmatrix} \succeq 0 \quad (3f)$$

In (3d), for those junctions with natural gas resources the upper and lower limits will be the minimum and maximum capacity of the natural gas resources. For those junction without natural gas resources, the upper and lower limits are zero. In the relaxed formulation, the employed lifting variables associated with each junction will form a sparse moment matrix associated with that junction as shown in (3f).

The problem (3) is a semi-definite programming (SDP) problem which represents the first order moment relaxation [7]. Although the presented convex relaxation is computationally inexpensive, the procured solution of the relaxed problem is exact and feasible for the original non-convex problem if and only if the first order moment matrices are rank-1. Otherwise, increasing the order of the moment relaxation tightens the relaxation and ensures the exactness of the procured solution [7]. However, increasing the order of the moment relaxation and the size of the moment matrix at this step will increase the computation burden exponentially and may not be tractable as the size of the network increases. Therefore, for the constraints associated with the moment matrices with a higher than one rank, the reformulation-linearization technique (RLT) is leveraged to reduce their rank. An example of such RLT constraints is given in (4) for the lower bound of the inequality given in (3d).

$$d_j^2 + 2d_j \textstyle\sum_{o\in J_j}\left(C_{j,o}\left(\gamma_{\sqrt{|\pi_j-\pi_o|}} - 2\gamma_{\overline{u}_{j,o}\sqrt{|\pi_j-\pi_o|}}\right)\right) + \sum_{o\in J_j}\left(C_{j,o}^2\left(\pi_j - \pi_o + 2\gamma_{-\overline{u}_{j,o}(\pi_j-\pi_o)}\right)\right) - \sum_s 2A_j^s \cdot \underline{v}_s \cdot \left(d_j + \sum_{o\in J_j} C_{j,o}\left(\gamma_{\sqrt{|\pi_j-\pi_o|}} - 2\gamma_{\overline{u}_{j,o}\sqrt{|\pi_j-\pi_o|}}\right)\right) + \sum_s A_j^s \cdot \underline{v}_s^2 \geq 0 \quad (4)$$

If solving the relaxed problem with RLT constraints renders rank-1 solution for all sparse moment matrices, the procured convex relaxation is exact. Otherwise, the next step is to add valid constraints to tighten the relaxed problem. An example of such valid constraints that leverage McCormick envelopes is given in (5).

$$\pi_j - \pi_o + 2\gamma_{-\overline{u}_{j,o}(\pi_j-\pi_o)} \geq 2\left(\sqrt{\overline{\pi} - \underline{\pi}}\right)\gamma_{\sqrt{|\pi_j-\pi_o|}} - \left(\overline{\pi} - \underline{\pi}\right) \quad (5a)$$

$$\pi_j - \pi_o \leq \gamma_{\overline{u}_{j,o}}\left(\overline{\pi} - \underline{\pi}\right) \quad (5b)$$

$$\pi_j - \pi_o + 2\gamma_{-\overline{u}_{j,o}(\pi_j-\pi_o)} \leq \left(\sqrt{\overline{\pi} - \underline{\pi}}\right)\gamma_{\sqrt{|\pi_j-\pi_o|}} - \left(\overline{\pi} - \underline{\pi}\right) \quad (5c)$$

If adding the valid constraints to the relaxed problem renders a rank-1 solution, the procured solution is exact. Otherwise, a higher order of the moment relaxation is employed for the matrices with higher than one rank. The asymptotic convergence is guaranteed with higher order of moment relaxation [7]. The procedure of adding RLT constraints, valid constraints, and higher order moment relaxation continues until the rank of all moment matrices is one. The proposed approach procures a rank-1 solution by tightening the relaxation using valid constraints and lower order of moment matrices. Otherwise, a higher-order moment relaxation is used to the extent necessary.

## IV. Illustrative Example

The characteristics of the natural gas resources and pipelines of a sample 6-junction network shown in Fig. 1, are given in Tables I and II, respectively. The minimum and maximum nodal pressures are 105 and 170 Psig respectively. The natural gas demands at junctions J1, J2, J3 are 2200, 650 and 1500 kcf

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respectively. The maximum compression ratio between junctions J2 and J7 is 1.1

TABLE I
NATURAL GAS RESOURCE CHARACTERISTICS

| Resource | Connected Junction | Minimum Capacity (kcf) | Maximum Capacity (kcf) | Cost ($/kcf) |
|---|---|---|---|---|
| $v1$ | J5 | 1500 | 5000 | 2.6 |
| $v2$ | J6 | 1000 | 6000 | 3.2 |

TABLE II
NATURAL GAS PIPELINE CHARACTERISTICS

| Line ID | Upstream Junction | Downstream Junction | Pipeline Constant (kcf /Psig) | Capacity (kcf) |
|---|---|---|---|---|
| PL1 | J2 | J1 | 75.5 | 3000 |
| PL2 | J4 | J2 | 91.5 | 1000 |
| PL3 | J4 | J3 | 82 | 3000 |
| PL4 | J5 | J4 | 72 | 5000 |
| PL5 | J7 | J6 | 81.5 | 6000 |

To solve the formulated SDP problem, Mosek 7 solver is employed. In this case, the volume of natural gas withdrawn from resource $v_1$ and $v_2$ is 2500 kcf and 1850 kcf. The procured pressures at the junctions of the network using the proposed convex relaxed formulation are compared with those procured by formulating SDP relaxed and MILP problems [2] in Table III. Here, the total operating cost is $12420, and the solution time is 0.87 sec. Using the SDP relaxed formulation, the total operation cost is $11910, and the solution time is 0.81 sec. Moreover, formulating the problem as MILP problem and solving using branch and bound approach leads to $12425 in operation cost and 7.23 sec in solution time. It is worth noting that the solution procured by solving the SDP relaxed problem renders the lower bound for the solution to the original nonconvex optimization problem and is not necessarily feasible for this problem. Moreover, the solution procured for the MILP problem is locally optimal.

The natural gas flow in the pipelines is compared for the solution of the proposed formulation, with the solutions of the SDP relaxed and MILP formulation in Table IV. In addition to the optimality gap, the procured flows may not be accurate for the MILP problem formulation because of the error in the piece-wise linearization of the Weymouth constraint. Here, the flow of natural gas in the pipelines procured by solving the SDP relaxed problem is not feasible for the original non-convex optimization problem. However, the proposed tight convex relaxation provides accurate and optimal junction pressure and flow in the pipelines that satisfy the Weymouth constraint.

TABLE III
NATURAL GAS PRESSURE AT JUNCTIONS USING DIFFERENT FORMULATIONS [PSIG]

| Junction ID | J1 | J2 | J3 | J4 | J5 | J6 | J7 |
|---|---|---|---|---|---|---|---|
| Proposed relaxation | 141.8 | 144.7 | 144 | 145.1 | 149.2 | 141.4 | 143.2 |
| SDP relaxation | 132.0 | 145.3 | 126.8 | 138.9 | 156.5 | 141.9 | 152.4 |
| MILP [2] | 110.0 | 113.8 | 112.8 | 114.3 | 119.4 | 113.8 | 116 |

TABLE IV
PROCURED NATURAL GAS FLOW WITHIN PIPELINES USING DIFFERENT FORMULATIONS [KCF]

| Pipeline ID | PL1 | PL2 | PL3 | PL4 | PL5 | PL6 |
|---|---|---|---|---|---|---|
| Proposed relaxation | 2200 | 1000 | 1500 | 2500 | 1850 | 1850 |
| SDP relaxation | 2200 | 1850 | 1500 | 3350 | 1000 | 1000 |
| MILP [2] | 2200 | 992.4 | 1500 | 2492.4 | 1857.4 | 1857.4 |

As the proposed tight convex relaxation seeks for a rank-1 solution, it is important to measure the tightness of the proposed convex relaxed formulation. Here, the average ratio of the largest to the second largest eigenvalues of the sparse SDP matrices associated with each junction is considered a measure of tightness. The ratio is increased from 2175 for the SDP relaxed formulation to 1.2E+9 for the proposed tight convex relaxed formulation. The relatively small tightness ratio of the SDP relaxed problem leads to an infeasible solution for the original non-convex problem, while the large tightness ratio of the proposed convex relaxed problem formed by applying the proposed relaxation approach leads to a feasible solution for the original non-convex problem.

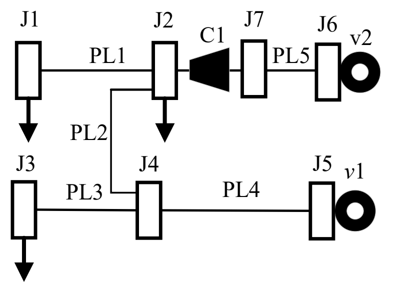

Fig. 1. The sample natural gas network

## V. CONCLUSION

This letter explored the non-convexity associated with the natural gas operational planning problem raised by the employment of Weymouth constraint. A methodology is proposed to procure a convex relaxation for the natural gas operation planning problem. The proposed approach will tighten the relaxation in order to procure a rank-1 solution of the relaxed problem that ensures feasibility of procured solution for the original non-convex problem. The effectiveness of the proposed methodology is verified in an example. The proposed convex relaxation can be further employed to address natural gas flow dynamics for the short-term operation problems.

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